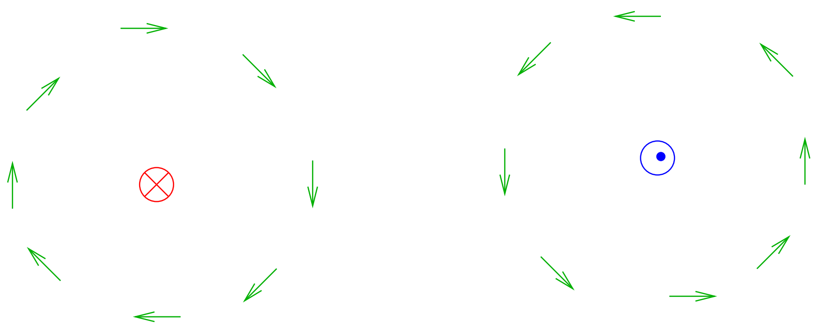
Magnetostatics

Electric charges are source of electric fields. An electric field exerts force on an electric charge, whether the charge happens to be moving or at rest.
One could similarly think of a magnetic charge as being the source of a magnetic field. However, isolated magnetic charge (or magnetic monopoles) have never been found to exist. Magnetic poles always occur in pairs (dipoles) – a north pole and a south pole. Thus, the region around a bar magnet is a magnetic field. What characterizes a magnetic field is the qualitative nature of the force that it exerts on an **electric charge**. The field does not exert any force on a static charge. However, if the charge happens to be moving (excepting in a direction parallel to the direction of the field) it experiences a force in the magnetic field.
It is not necessary to invoke the presence of magnetic poles to discuss the source of magnetic field. Experiments by Oersted showed that a magnetic needle gets deflected in the region around a current carrying conductor. The direction of deflection is shown in the figure below.



Current into the page

Current out of the page

Thus a current carrying conductor is the source of a magnetic field. In fact, a magnetic dipole can be considered as a closed current loop.

Lorentz Force :

We know that an electric field $\vec{E}$ exerts a force $q\vec{E}$ on a charge q. In the presence of a magnetic field $\vec{B}$, a charge q experiences an additional force

$$\vec{F}_m = q\vec{v} \times \vec{B}$$

where $\vec{v}$ is the velocity of the charge. Note that

- There is no force on a charge at rest.
- A force is exerted on the charge only if there is a component of the magnetic field perpendicular to the direction of the velocity, i.e. *the component of the magnetic field parallel to $\vec{v}$ does not contribute to $\vec{F}_m$.*
- $\vec{v} \cdot \vec{F}_m = 0$, which shows that the magnetic force does not do any work.

In the case where both $\vec{E}$ and $\vec{B}$ are present, the force on the charge q is given by

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

This is called **Lorentz force** after H.E. Lorentz who postulated the relationship. It may be noted that the force expression is valid even when $\vec{E}$ and $\vec{B}$ are time dependent.

Unit of Magnetic Field

From the Lorentz equation, it may be seen that the unit of magnetic field is Newton-second/coulomb-meter, which is known as a Tesla (T). (The unit is occasionally written as Weber/m^2 as the unit of magnetic flux is known as Weber). However, Tesla is a very large unit and it is common to measure $\vec{B}$ in terms of a smaller unit called Gauss,

$$1\text{T} = 10^4 \text{ G}$$

It may be noted that $\vec{B}$ is also referred to as magnetic field of induction or simply as the induction field. However, we will use the term "magnetic field".

Motion of a Charged Particle in a Uniform Magnetic Field

Let the direction of the magnetic field be taken to be z- direction,

$$\vec{B} = B\hat{k}$$

we can write the force on the particle to be

$$\vec{F}_m = m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}$$

The problem can be looked at qualitatively as follows. We can resolve the motion of the charged particle into two components, one parallel to the magnetic field and the other perpendicular to it. Since the motion parallel to the magnetic field is not affected, the velocity component in the z-direction remains constant.

$$v_z(t) = v_z(t = 0) = u_z$$

where $\vec{u}$ is the initial velocity of the particle. Let us denote the velocity component perpendicular to the direction of the magnetic field by $v_\perp$. Since the force (and hence the acceleration) is perpendicular to the direction of velocity, the motion in a plane perpendicular to $\vec{B}$ is a circle. The centripetal force necessary to sustain the circular motion is provided by the Lorentz force

$$\frac{mv_\perp^2}{R} = \mid q \mid v_\perp B$$

where the radius of the circle R is called the Larmor radius, and is given by

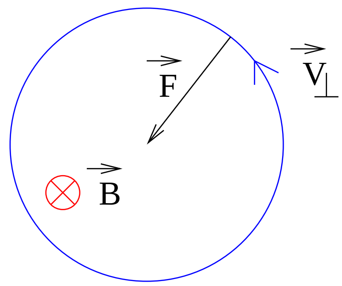
$$R = \frac{mv_\perp}{\mid q \mid B}$$

The time taken by the particle to complete one revolution is

$$T = \frac{2\pi R}{v_\perp}$$

The cyclotron frequency ω_c is given by

$$\omega_c = \frac{2\pi}{T} = \frac{\mid q \mid B}{m}$$



⊗ $\vec{B}$ Magnetic field into the page.
Figure shows directions of force and velocity for a positive charge.

Motion in a Magnetic Field – Quantitative

Let the initial velocity of the particle be $\vec{u}$. we may take the direction of the component of $\vec{u}$ perpendicular to $\vec{B}$ as the $x-$ direction, so that

$$\vec{u} = (u_x, 0, u_z)$$

Let the velocity at time t be denoted by $\vec{v}$

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

we can express the force equation in terms of its cartesian component

$$\begin{aligned} m\frac{dv_x}{dt} = q(v_yB_z - v_zB_y) &= qv_yB \\ m\frac{dv_y}{dt} = q(v_zB_x - v_xB_z) &= -qv_xB \\ m\frac{dv_z}{dt} = q(v_xB_y - v_yB_x) &= 0 \end{aligned}$$

where we have used $B_x = B_y = 0$ and $B_z = B$.
The last equation tells us that no force acts on the particle in the direction in which $\vec{B}$ acts, so that

$$v_z = \text{constant } = u_z$$

The first two equations may be solved by converting them into second order differential equatons. This is done by differentiating one of the equations with respect to time and substituting the other equation in the resulting second order equation. For instance, the equation for v_x is given by

$$\frac{d^2 v_x}{dt^2} = \frac{qB}{m}\frac{dv_y}{dt} = -\frac{q^2B^2}{m^2}v_x$$

The equation is familiar in the study of simple harmonic motion. The solutions are combination of sine and cosine functions.

$$v_x = A\sin\omega_c t + B\cos\omega_c t$$

where

$$\omega_c = \frac{qB}{m}$$

is called the *cyclotron frequency* and A and B are constants. These constants have to be determined from initial conditions. By our choice of x and y axes, we have

$$v_x(t = 0) = u_x$$

so that $B = u_x$. Differentiating the above equation for u_x,

$$\begin{aligned} \frac{dv_x}{dt} &= A\omega_c\cos\omega_c t - B\omega_c\sin\omega_c t \\ &\equiv \frac{qB}{m}v_y = \omega_c v_y \end{aligned}$$

Since $v_y = 0$ at $t = 0$, we have $A = 0$. Thus, the velocity components at time t are given by

$$\begin{aligned} v_x &= u_x\cos\omega_c t \\ v_y &= -u_x\sin\omega_c t = u_x\sin(\omega_c t + \frac{\pi}{2}) \end{aligned}$$

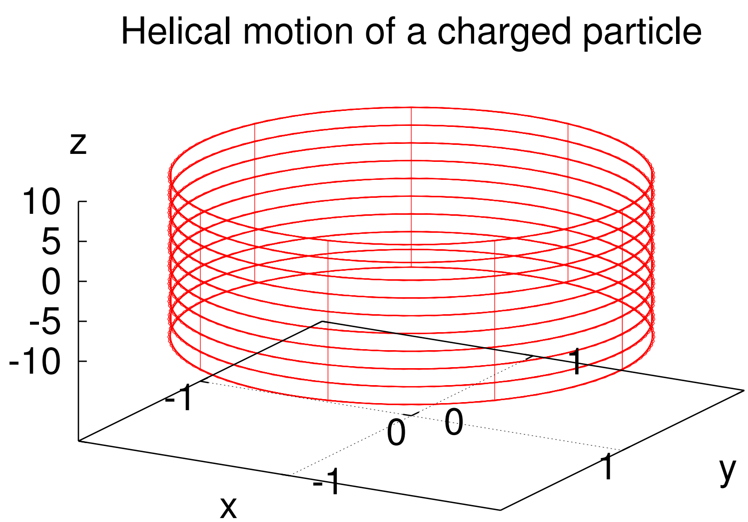
which shows that v_x and v_y vary harmonically with time with the same amplitude but with a phase difference of $\pi/2$. Equation of the trajectory may be obtained by integrating the equations for velocity components

$$x(t) = \frac{u_x}{\omega_c}\sin\omega_c t + x_0$$
$$y(t) = \frac{u_x}{\omega_c}\cos\omega_c t + y_0$$
$$z(t) = u_z t + z_0$$

where x_0, y_0 and z_0 are constants of integration representing the initial position of the particle. The equation to the projection of the trajectory in the x-y plane is given by

$$(x - x_0)^2 + (y - y_0)^2 = \frac{u_x^2}{\omega_c^2}$$

which represents a circle of radius u_x/ω_c, centered about (x_o, y_0). As the z- component of the velocity is constant, the trajectory is a helix.



A plot of the motion of a charged particle in a constant magnetic field.

Motion in a crossed electric and magnetic fields

The force on the charged particle in the presence of both electric and magnetic fields is given by

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Let the electric and magnetic fields be at right angle to each other, so that,

$$\vec{E} \cdot \vec{B} = 0$$

If the particle is initially at rest no magnetic force acts on the particle. As the electric field exerts a force on the particle, it acquires a velocity in the direction of $\vec{E}$. The magnetic force now acts sidewise on the particle.
For a quantitative analysis of the motion, let $\vec{E}$ be taken along the x-direction and $\vec{B}$ along z-direction. As there is no component of the force along the z-direction, the velocity of the particle remains zero in this direction. The motion, therefore, takes place in x-y plane.
The equations of motion are

$$m\frac{dv_x}{dt} = q(\vec{E} + \vec{v} \times \vec{B})_x = qE + qBv_y \quad (1)$$

$$m\frac{dv_y}{dt} = q(\vec{E} + \vec{v} \times \vec{B})_y = -qBv_x \quad (2)$$

As in the earlier case, we can solve the equations by differentiating one of the equations and substituting the other,

$$m\frac{d^2v_x}{dt^2} = qB\frac{dv_y}{dt} = -\frac{q^2B^2}{m}v_x$$

which, as before, has the solution

$$v_x = A\sin\omega_c t$$

with $\omega_c = qB/m$. Substituting this solution into the equation for v_y, we get, using Eqn. (1)

$$\begin{aligned} v_y &= -\frac{E}{B} + \frac{m}{qB}\frac{dv_x}{dt} \\ &= -\frac{E}{B} + \frac{m}{qB}A\omega_c\cos\omega_c t \end{aligned}$$

Since $v_y = 0$ at $t = 0$, the constant $A = Eq/m\omega_c = E/B$, so that Thus we have

$$\begin{aligned} v_x &= \frac{E}{B}\sin\omega_c t \\ v_y &= \frac{E}{B}(\cos\omega_c t - 1) \end{aligned}$$

The equation to the trajectory is obtained by integrating the above equation and determining the constant of integration from the initial position (taken to be at the origin),

$$\begin{aligned} x &= \frac{E}{B\omega_c}(1 - \cos\omega_c t) \\ y &= \frac{E}{B\omega_c}(\sin\omega_c t - \omega_c t) \end{aligned}$$

The equation to the trajectory is

$$\left(x - \frac{E}{B\omega_c}\right)^2 + \left(y - \frac{Et}{B}\right)^2 = \frac{E^2}{B^2\omega_c^2}$$

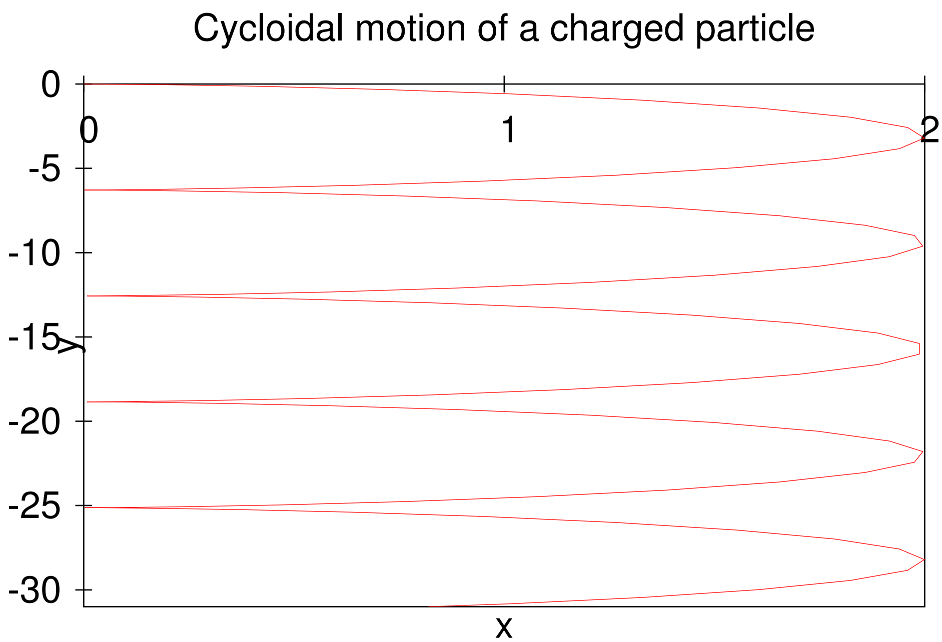
which represents a circle of radius

$$R = \frac{E}{B\omega_c}$$

whose centre travels along the negative y direction with a constant speed

$$v_0 = \frac{E}{B}$$

The trajectory resembles that of a point on the circumference of a wheel of radius R, rolling down the y-axis without slipping with a speed v_0. The trajectory is known as a cycloid.



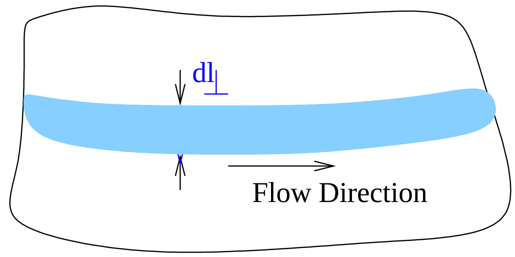
Exercise : Find the maximum value of x attained by the particle during the cycloidal motion and determine the speed of the particle at such points. (Ans. $x_{max} = 2E/B\omega_c$ speed $2E/B$.)

Current Density :

The current in a wire is a measure of the amount of charge flowing through any point of the wire in unit time. If the charge density in the wire is λ, the current is given by $I = \lambda v$ where v is the drift velocity of the charge carriers. The carriers travel a distance $dl = vdt$ in time dt so that the amount of charge that flows in time dt through any point of the wire is $dQ = Idt = \lambda vdt$. The unit of current is Coulomb per second, known as Ampere.

When we consider motion of charges on a surface or in a volume, the charges move in all directions and we have to consider average velocities. We need to introduce the concept of a charge sensity which is then a vector quantiy at a point in the material. For instance, consider charges flowing two dimension.

If we consider a ribbon along the direction of flow in the surface, we could talk of the amount of charge flowing past a length $dl_\perp$ normal to the direction of flow at a point. The surface current density that we talk of at this point is the **current per unit length** oriented perpendicular to the flow at this point.



Thus the current density is

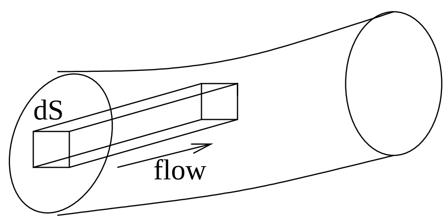
$$K = \frac{dI}{dl_\perp}$$

The unit obviously is Ampere per meter. If σ is the surface charge density and v is the average velocity, then the amount of charge flowing past the length element in time dt is clearly $dQ = \sigma dl_\perp v dt$. The current flowing past the length element is $dQ/dt = \sigma dl_\perp v$ so that the current density is $K = \sigma v$.
The Lorentz force on charges in the surface is

$$\vec{F}_m = \int (\sigma da)\vec{v} \times \vec{B} = \int (\vec{K} \times \vec{B}) da$$

.

In three dimensions, the concept is very similar. One has to consider the current flowing through a surface element oriented perpendicular to the direction of flow.



The volume current density $\vec{J}$ is the defined as

$$\vec{J} = \frac{dI}{dS_\perp}$$

which has a direction perpendicular to the surface element but going in the direction of flow. The Lorentz force acting on the charges in the volume is then given by

$$\vec{F}_m = \int (\rho d\tau)\vec{v} \times \vec{B} = \int (\vec{J} \times \vec{B}) d\tau$$

Equation of Continuity :

The current through any closed volume is thus given by

$$I = \int_S J dS_\perp = \int_S \vec{J} \cdot d\vec{S}$$

where S is the surface bounding the volume. Using divergence theorem, we may conver the surface integral to a volume integral over the divergence, so that

$$I = \int \nabla \cdot \vec{J} d\tau$$

We know that $\nabla \cdot \vec{J}$ gives the *outward* flux through the surface. The net outward flux must result in a net decrease of charges within the volume. In other words, we must have,

$$\int \nabla \cdot \vec{J} d\tau = -\frac{d}{dt} \int \rho d\tau = -\int \frac{\partial \rho}{\partial t} d\tau$$

As the relation is true for arbitrary volume, we have the equation of continuity

$$\boxed{\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

In magnetostatics, we study effects of what are known as **steady current**. The word steady in thius context does not mean stationary because current itself implies that charges are moving in time. Steady implies that the rate of flow is constant so that $\partial\rho/\partial t = 0$. For steady currents, the equation of continuity becomes

$$\boxed{\nabla \cdot \vec{J} = 0}$$

Biot- Savarts' Law

We have seen that electrostatics was formulated on the basis of empirical law of Coulomb. The corresponding law for magnetostatics is Biot-Savart's law. which provides an expression for the magnetic field due to a current segment. The field $d\vec{B}$ at a position $\vec{r}$ due to a current segment $I\vec{dl}$ is experimentally found to be perpendicular to $\vec{dl}$ and $\vec{r}$. The magnitude of the field is

- proportional to the length $\mid dl \mid$ and to the current I and to the sine of the angle between $\vec{r}$ and $\vec{dl}$.
- inversely proportional to the square of the distance r of the point P from the current element.

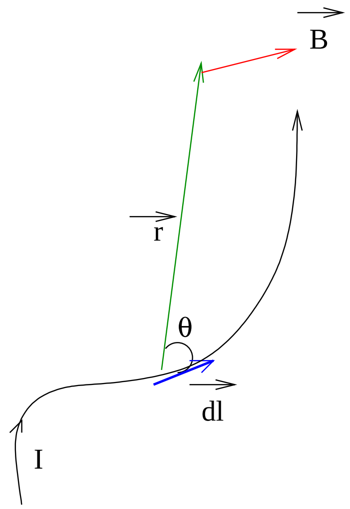
Mathematically,

$$d\vec{B} \propto I \frac{\vec{dl} \times \hat{r}}{r^2}$$

In SI units the constant of proportionality is $\mu_0/4\pi$, where μ_0 is the permeability of the free space. The value of μ_0 is

$$\mu_0 = 4\pi \times 10^{-7} \text{ N/amp}^2$$

The expression for field at a point P having a position vector $\vec{r}$ with respect to the current element is



$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{\vec{dl} \times \hat{r}}{r^2}$$

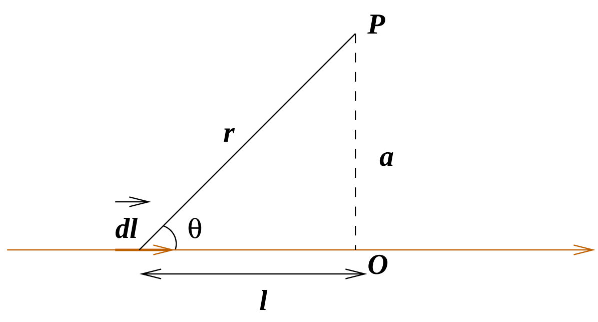
For a conducting wire of arbitrary shape, the field is obtained by vectorially adding the contributions due to such current elements as per superposition principle,

$$\vec{B}(P) = \frac{\mu_0}{4\pi} I \int \frac{\vec{dl} \times \hat{r}}{r^2}$$

where the integration is along the path of the current flow.

Example 1 : Field due to a straight wire carrying current

The direction of the field at P due to a current element $\vec{dl}$ is along $\vec{dl} \times \vec{r}$, which is a vector normal to the page (figure on the left) and coming out of it.



We have,

$$\frac{\vec{dl} \times \hat{r}}{r^2} = \frac{\mid dl \sin\theta \mid}{r^2} \hat{k}$$

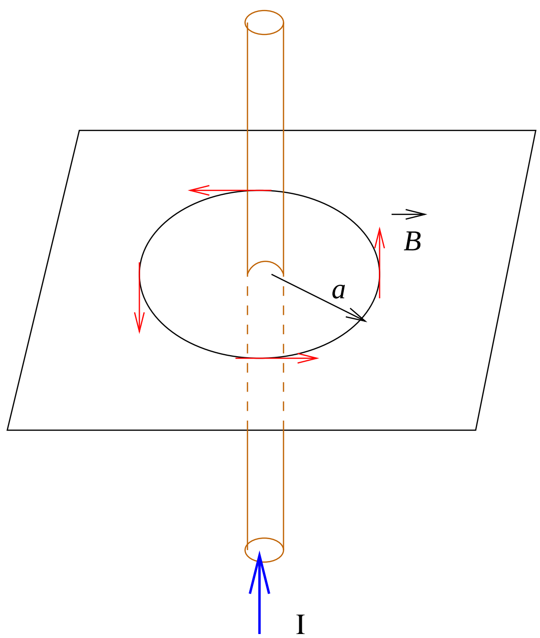
where the plane of the figure is taken as the x-y plane and the direction of outward normal is parallel to z-axis. If a be the distance of the oint P from the wire, we have

$$\begin{aligned} r &= a/\sin\theta \\ l &= a\cot\theta \\ dl &= -(a/\sin^2\theta)d\theta \end{aligned}$$

Thus

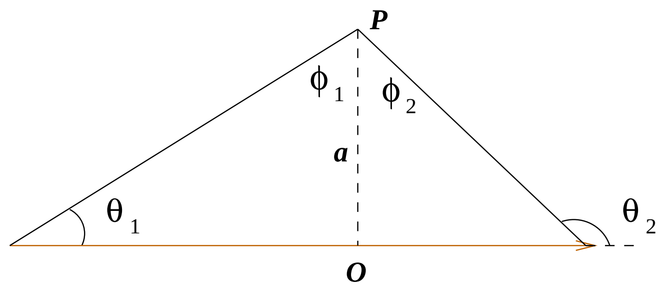
$$\frac{\vec{dl} \times \hat{r}}{r^2} = \frac{\sin\theta}{a} d\theta\hat{k}$$

The direction of the magnetic field at a distance a from the wire is tangential to a circle of radius a,as shown.



Since the magnetic field due to all current elements at P are parallel to the z-direction, the field at P due to a wire, the ends of which make angles ϕ_1 and ϕ_2 at P is given by a straightforward integration

$$\begin{aligned}\vec{B} &= \frac{\mu_0 I}{4\pi a}\int_{\theta_1}^{\theta_2} \sin\theta d\theta \hat{k} \\ &= \frac{\mu_0 I}{4\pi a}\left[-\cos\theta\right]_{\theta_1}^{\theta_2} \hat{k} \\ &= \frac{\mu_0 I}{4\pi a}\left[-\cos\theta_1 - \cos\theta_2\right] \hat{k} \\ &= \frac{\mu_0 I}{4\pi a}\left[\sin\phi_1 + \sin\phi_2\right] \hat{k}\end{aligned}$$



Note that both the angles ϕ_1 and ϕ_2 are acute angles.
If we consider an infinite wire (also called long straight wire), we have $\phi_1 = \phi_2 = \pi/2$, so that the field due to such a wire is

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{k}$$

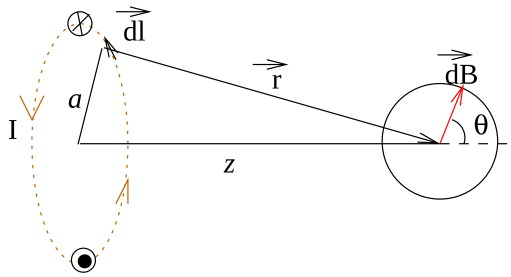
where the direction of the field is given by the Right hand rule.

Exercise :
A conductor in the shape of an n-sided polygon of side a carries current I. Calculate the magnitude of the magnetic field at the centre of the polygon. [Ans. $(\mu_0 In/\pi a)\sin(\pi/n)$.]

Example 2 : Field due to a circular coil on its axis
Consider the current loop to be in the x-y plane, which is taken perpendicular to the plane of the paper in which the axis to the loop (z-axis) lies. Since all length elements on the circumference of the ring are perpendicular to $\vec{r}$, the magnitude of the field at a point P is given by

$$dB = \frac{\mu_0 I}{4\pi}\frac{dl}{r^2}$$

The direction of the field due to every element is in the plane of the paper and perpendicular to $\vec{r}$, as shown. Corresponding to every element $\vec{dl}$ on the circumference of the circle, there is a diametrically opposite element which gives a magnetic field $d\vec{B}$ in a direction such that the component of $d\vec{B}$ perpendicular to the axis cancel out in pairs.



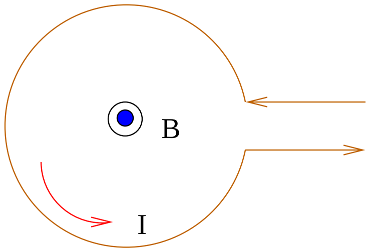
The resultant field is parallel to the axis, its direction being along the positive z-axis for the current direction shown in the figure. The net field is

$$\begin{aligned}\vec{B} &= \frac{\mu_0 I}{4\pi}\int \frac{dl}{r^2}\cos\theta \\ &= \frac{\mu_0 I}{4\pi}\frac{\cos\theta}{r^2}\int dl \\ &= \frac{\mu_0 I\cos\theta}{4\pi r^2}2\pi a = \frac{\mu_0 I a\cos\theta}{2r^2}\end{aligned}$$

In terms of the distance z of the point P and the radius a, we have

$$B = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}}$$

The direction of the magnetic field is determined by the following Right Hand Rule.



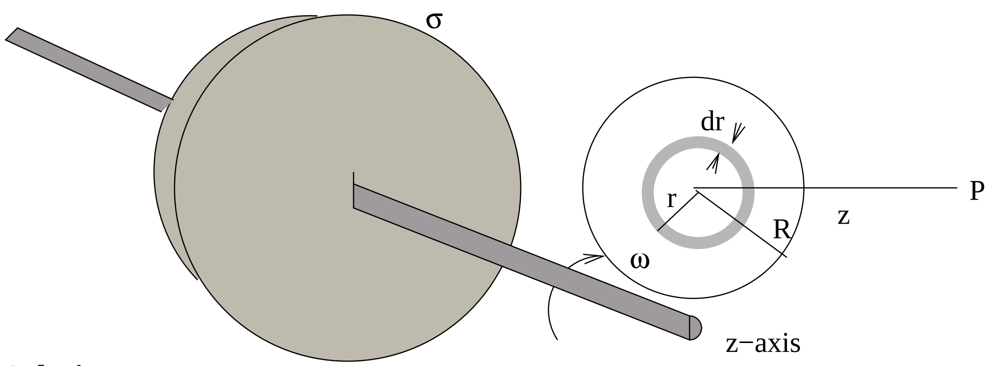
If the palm of the right hand is curled in the direction of the current, the direction in which the thumb points gives the direction of the magnetic field at the centre of the loop. The field is, therefore, outward in the figure shown.

Note that for $z \gg a$, i.e. the field due to circular loop at large distances is given by

$$B = \frac{\mu_0 I a^2}{2z^3} = \frac{\mu_0 \mu}{2\pi z^3}$$

where $\mu = I\pi a^2$ is the magnetic moment of the loop. The formula is very similar to the field of an electric dipole. Thus a current loop behaves like a magnetic dipole.

Example 3 :
A thin plastic disk of radius R has a uniform surface charge density σ. The disk is rotating about its own axis with an angular velocity ω. Find the field at a distance z along the axis from the centre of the disk.



Solution :
The current on the disk can be calculated by assuming the rotating disk to be equivalent to a collection of concentric current loops. Consider a ring of radius r and of width dr. As the disk is rotating with an angular speed ω, the rotating charge on the ring essentially behaves like a current loop carrying current $\sigma \cdot 2\pi r dr \cdot \omega/2\pi = \sigma\omega r dr$.
The field at a distance z due to this ring is

$$dB = \frac{\mu_0(\sigma\omega r dr)}{2}\frac{r^2}{(r^2+z^2)^{3/2}}$$

The net field is obtained by integrating the above from $r = 0$ to $r = R$

$$\begin{aligned} B &= \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3}{(r^2+z^2)^{3/2}}dr \\ &= \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^2+z^2-z^2}{(r^2+z^2)^{3/2}}r dr \end{aligned}$$

The integral above may easily be evaluated by a substitution $x = r^2 + z^2$. The result is

$$B = \mu_0\sigma\omega\left[\frac{R^2+2z^2}{(R^2+z^2)^{3/2}} - 2z\right]$$

The field at the centre of the disk ($z = 0$) is

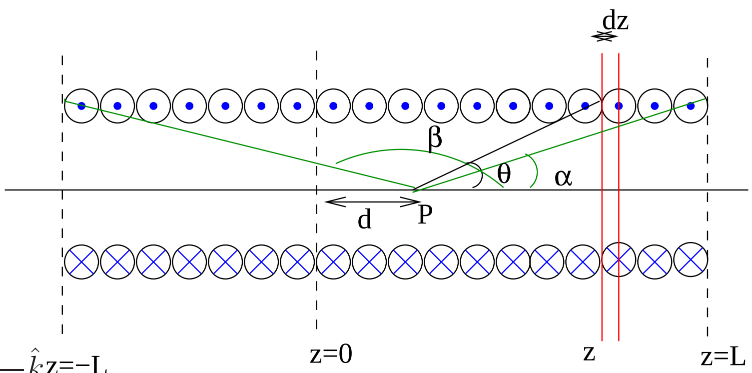
$$B(z=0) = \mu_0\sigma\omega R$$

Exercise : Find the magnetic moment of the rotating disk of Example 7. [Ans. $\pi\omega R^4/4$]

Example 4 : Field of a solenoid on its axis

Consider a solenoid of N turns. The solenoid can be considered as stacked up circular coils. The field on the axis of the solenoid can be found by superposition of fields due to all circular coils. Consider the field at P due to the circular turns between z and $z + dz$ from the origin, which is taken at the centre of the solenoid. The point P is at $z = d$. If L is the length of the solenoid, the number of turns within z and $z + dz$ is $Ndz/L = ndz$, where n is the number of turns per unit length.

The magnitude of the field at P due to these turns is given by



$$dB = \frac{\mu_0 NIdz}{2L}\frac{a^2}{[a^2 + (z-d)^2]^{3/2}}$$

The field due to each turn is along $\hat{k}$; hence the fields due to all turns simply add up. The net field is

$$\vec{B} = \frac{\mu_0 NIa^2}{2L}\int_{-L/2}^{L/2}\frac{dz}{[a^2 + (z-d)^2]^{3/2}}\hat{k}$$

The integral above is easily evaluated by substituting

$$\begin{aligned} z - d &= a\cot\theta \\ dz &= -a\text{cosec}^2\theta d\theta \end{aligned}$$

The limits of integration on θ are α and β as shown in the figure. With the above substitution

$$\vec{B} = -\frac{\mu_0 nI}{2}\int_{\alpha}^{\beta}\sin\theta d\theta = \frac{\mu_0 nI}{2}(\cos\alpha - \cos\beta)\hat{k}$$

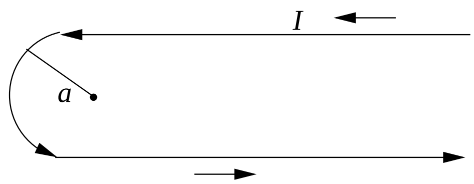
For a long solenoid, the field on the axis at points far removed from the ends of the solenoid may be obtained by substituting $\alpha = 0°$ and $\beta = 180°$, so that,

$$\vec{B} = \mu_0 nI\hat{k}$$

The field is very nearly constant. For points on the axis far removed from the ends but outside the solenoid, $\alpha \approx \beta$ so that the field is nearly zero.

Example 5 :

Determine the field at the point located at the centre P of the semi-circular section of the hairpin bend shown in the figure.



Solution :

The field at P may be determined by superposition of fields due to the two straight line sections and the semicircular arc. The contribution due to all three sections add up as the field due to each is into the plane of the paper.

The field due to each straight line section is obtained by putting $\phi_1 = 90°$ and $\phi_2 = 0°$ in the expression obtained in Example 5 above. The field due to each wire is $\mu_0 I/4\pi a$.

For the semi-circular arc, each length element on the circumference is perpendicular to $\vec{r}$, the vector from the length element to the point P. Thus

$$\begin{aligned} B_{arc} &= \frac{\mu_0 I}{4\pi}\int\frac{\vec{dl}\times\hat{r}}{r^2} \\ &= \frac{\mu_0 I}{4\pi a^2}\int dl = \frac{\mu_0 I}{4\pi a^2}\cdot\pi a = \frac{\mu_0 I}{4a} \end{aligned}$$

The net field due to the current in the hairpin bend at P is

$$B = \frac{\mu_0 I}{2\pi a} + \frac{\mu_0 I}{4a}$$

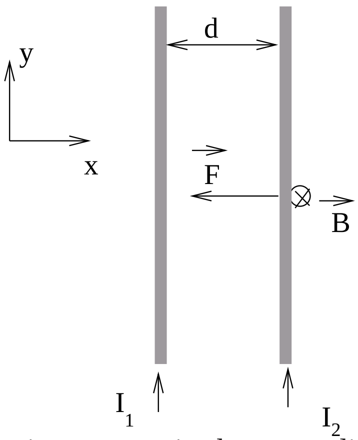
Example 6 : Force between two long and parallel wires.

Force due to the first wire at the position of the second wire is given by

$$\vec{B} = -\frac{\mu_0 I_1}{2\pi d}\hat{k}$$

where $\hat{k}$ is a unit vecor out of the page.

The force experienced by the second wire in this field is

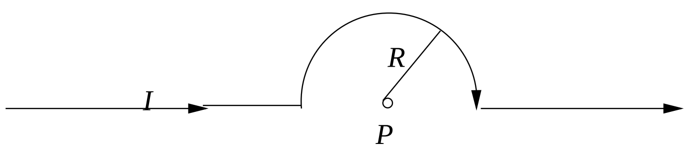


$$\vec{F} = \int (\vec{I_2} \times \vec{B}) dl$$
$$= -\frac{\mu_0 I_1 I_2}{2\pi d}(\hat{\jmath} \times \hat{k}) \int dl$$
$$= -\frac{\mu_0 I_1 I_2}{2\pi d}\hat{\imath} \int dl$$

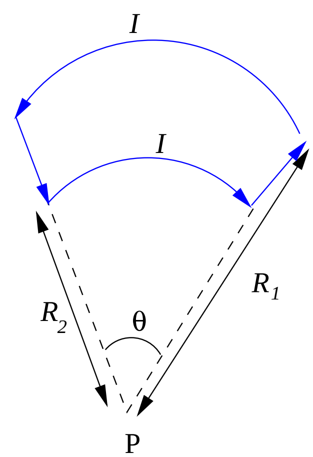
Thus the force between the wires carrying current in the same direction is attractive and is $\mu_0 I_1 I_2 / 2\pi d$ per unit length.

Exercise :

Determine the magnetic field at the point P for the two geometries shown in the figures below. [Ans . (a) $\mu_0 I/4R$ (b) $\frac{\mu_0 I (R_1 - R_2)\theta}{4\pi R_1 R_2}$]



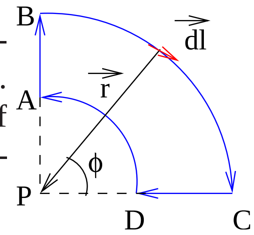
(a)



(b)

Example 7 :

Find the field at the centre of a conductor shaped as shown in the figure. The curved sections are quadrants of circles of radii a and b and the conductor carries a current I.



Solution :

For straight segments AB and CD $d\vec{l} \times \vec{r} = 0$ and there is no contribution to the magnetic field. For the arcs BC and DA use cylindrical coordinates. For instance, for the arc of radius b, we have $d\vec{l} = -bdl\phi\hat{\phi}$, the minus sign is because the current is in the clockwise direction, and $\vec{r} = -b\hat{\rho}$ so that

$$d\vec{l} \times \vec{r} = b^2 d\phi\hat{\phi} \times \hat{\rho} = -b^2 d\phi\hat{k}$$

Similarly, for the smaller arc, since the current is counterclockwise,

$$d\vec{l} \times \vec{r} = a^2 d\phi\hat{k}$$

where $\hat{k}$ is in the z-direction, i.e. perpendicular to the plane of the arcs. Using Biot-Savart's law, we get

$$\begin{aligned} \vec{B} &= \frac{\mu_0 I}{4\pi}\int_0^{\pi/2} d\phi \left(\frac{1}{a} - \frac{1}{b}\right) \\ &= \frac{\mu_0 I}{8}\hat{k}\left(\frac{1}{a} - \frac{1}{b}\right) \end{aligned}$$

Ampere's Law

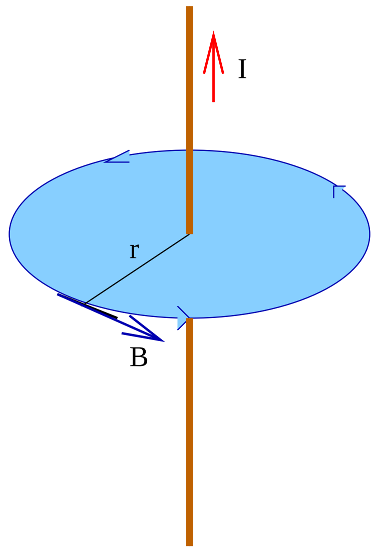
Biot-Savart's law for magnetic field due to a current element is difficult to visualize physically as such elements cannot be isolated from the circuit which they are part of. Andre Ampere formulated a law based on Oersted's as well as his own experimental studies. Ampere's law states that "*the line integral of magnetic field around any closed path equals μ_0 times the current which threads the surface bounded by such closed path.*. Mathematically,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed} \tag{1}$$

In spite of its apparent simplicity, Ampere's law can be used to calculate magnetic field of a current distribution in cases where a lot of information exists on the behaviour of $\vec{B}$. The field must have enough symmetry in space so as to enable us to express the left hand side of (1) in a functional form. The simplest application of Ampere' s law consists of applying the law to the case of an infinitely long straight and thin wire.

Example : Magnetic Field of a long wire

By symmetry of the problem we know that the magnitude of the field at a point can depend only on the distance of the point from the wire. Further, the field is tangential to the circle of radius r, its direction being given by the right hand rule.



Thus the integral around the circle is

$$\oint \vec{B} \cdot \vec{dl} = B \oint dl = B \cdot 2\pi r$$

Equating this to $\mu_0 I$, we get

$$B = \frac{\mu_0 I}{2\pi r}$$

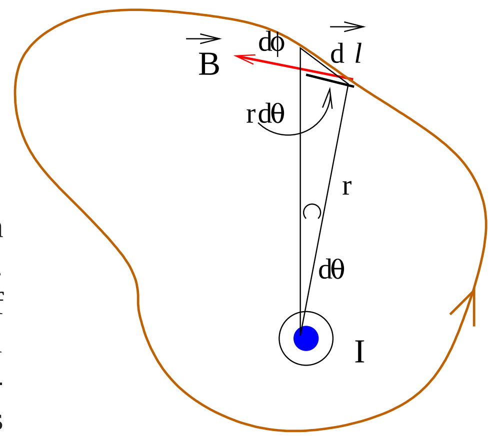
which is consistent with the result obtained from Biot-Savart's law.

Let us see if the result above is consistent with a path which is not circular, as shown in the figure. The field at every element $\vec{dl}$ of the path is perpendicular to $\vec{r}$. From geometry, it can be seen that

$$\vec{B} \cdot \vec{dl} = Bdl \cos\phi = Brd\theta$$

Thus

$$\oint \vec{B} \cdot \vec{dl} = \oint \frac{\mu_0 I}{2\pi r} r d\theta = \frac{\mu_0 I}{2\pi} \oint d\theta = \mu_0 I$$

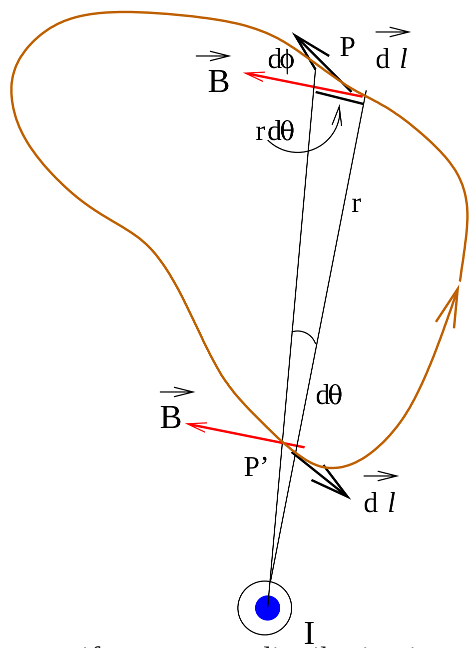


We need to specify the direction along which the path is traversed. This is done by Right Hand Rule. If we curl the fingers of our right hand along the path of integration, the direction along which the thumb points is the direction of current flow.

For the case where the path of integration lies totally outside the path of the current, for every element $\vec{dl}$ at P, there exists another element at P' for which $\vec{B} \cdot \vec{dl}$ has opposite sign. Thus when complete line integral is taken, the contributions from such pairs add to zero

$$\oint \vec{B} \cdot \vec{dl} = 0$$

Combining these, we get Ampere's law in the form of Eqn. (1)

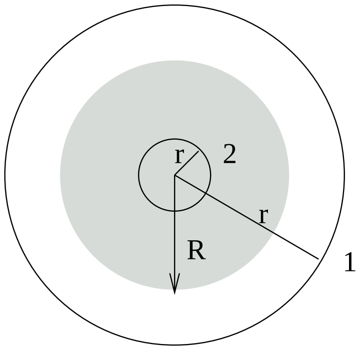


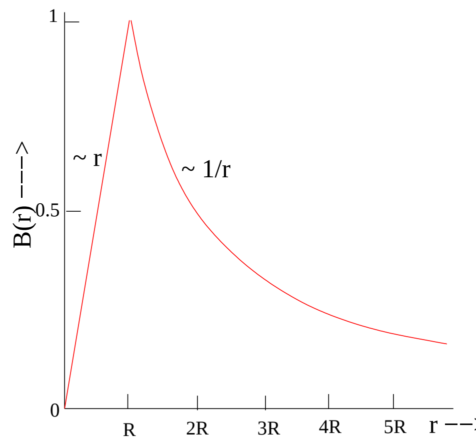
Example 8 : Calculate the field due to a uniform current distribution in an infinite wire of cross sectional radius R.

Solution :

Let the cross section of the wire be circular with a radius R. Take the current direction to be perpendicular to the page and coming out of it. Symmetry of the problem demands that the magnitude of the field at a point is dependent only on the distance of the point from the axis of the wire. Consider an amperian loop of radius r. As before we have

$$\oint \vec{B} \cdot \vec{dl} = 2\pi r B$$





If $r > R$ (as in loop 1), the entire current is enclosed by the loop. Hence $I_{enclosed} = I$ so that

$$B = \frac{\mu_0 I}{2\pi r}$$

If $r < R$ (loop 2), the current enclosed is proportional to the area, i.e.

$$I_{enclosed} = I\frac{\pi r^2}{\pi R^2} = I\frac{r^2}{R^2}$$

so that

$$B = \frac{\mu_o I}{2\pi R^2} r$$

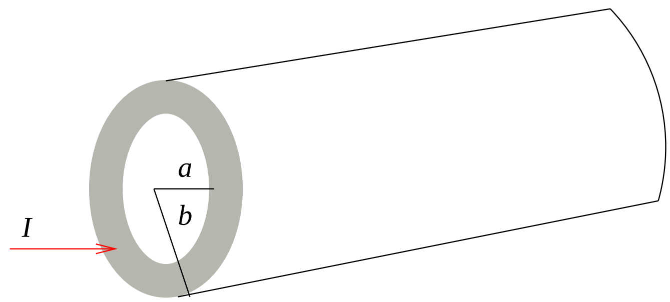
The field distribution with distance is as shown.

Exercise :

A long wire of cross sectional radius R carries a current I. The current density varies as the square of the distance from the axis of the wire. Find the magnetic field for $r < R$ and for $r > R$. (Hint : First show that the current density $J = 2Ir^2/\pi R^4$ and obtain an expression for current enclosed for $r < R$. Answer : $B = \mu_0 I/2\pi r$ for $r > R$ and $B = \mu_0 I r^3/2\pi R^4$ for $r < R$.)

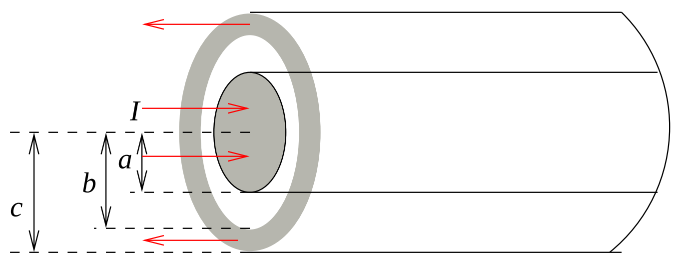
Exercise :

A hollow cylindrical conductor of infinite length carries uniformly distributed current I from $a < r < b$. Determine magnetic field for all r. (Answer : Field is zero for $r < a$, $B = \frac{\mu_0 I}{2\pi r}\frac{r^2 - a^2}{b^2 - a^2}$ for $a < r < b$ and $B = \mu_0 I/2\pi r$ for $r > b$.)



Exercise :

A coaxial cable consists of a solid conductor of radius a with a concentric shell of inner radius b and outer radius c. The space between the solid conductor and the shell is supported by an insulating material.

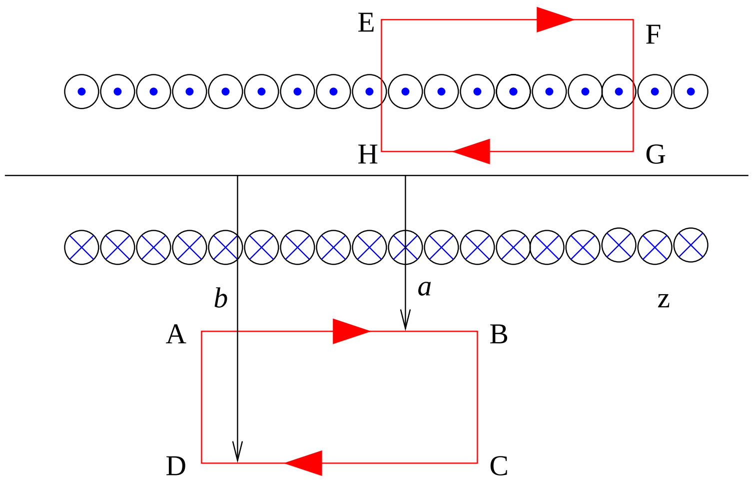


A current I goes into the inner conductor and is returned by the outer shell. Assume the cuurent densities to be uniform both in the shell and in the inner conductor. Calculate magnetic field everywhere. (Ans. $B = \mu_0 Ir/2\pi b^2$ inside the inner conductor, $B = \mu_0 I/2\pi r$ between the shell and the inner conductor, $B = \frac{\mu_0 I}{2\pi r}\frac{c^2 - r^2}{c^2 - b^2}$)

Exercise : Determine the magnetic field in a cylindrical hole of radius a inside a cylindrical conductor of radius b. The cylinders are of infinite length and their axes are parallel, being separated by a distance d. The conductor carries a current I of uniform density. (Hint : The problem is conveniently solved by imagining currents of equal and opposite densities flowing in the hole and using superposition principle to calculate the field. Answer : The field inside the hole is constant $B = \mu_0 Id/2\pi(b^2 - a^2)$)

Example 9 : Field of a long solenoid

We take the solenoid to be closely wound so that each turn can be considered to be circular. We can prove that the field due to such a solenoid is entirely confined to its interior, i.e. the field outside is zero, To see this consider a rectangular amperian loop parallel to the axis of the solenoid.



Field everywhere on AB is constant and is $B(a)$. Likewise the field everywhere on CD is $B(b)$. By Right hand rule, the field on AB is directed along the loop while that on CD is oppositely directed. On the sides AD and BC, the magnetic field direction is perpendicular to the length element and hence $\vec{B} \cdot \vec{dl}$ is zero everywhere on these two sides. Thus

$$\oint \vec{B} \cdot \vec{dl} = L(B(a) - B(b))$$

By Ampere's law, the value of the integral is zero as no current is enclosed by the loop. Thus $B(a) = B(b)$. The field outside the solenoid is, therefore, independent of the distance from the axis of the solenoid. However, from physical point of view, we expect the field to vanish at large distances. Thus $B(a) = B(b) = 0$.
To find the field inside, take an amperian loop EFGH with its length parallel to the axis as before, but with one of the sides inside the solenoid while the other is outside. The only contribution to $\oint \vec{B} \cdot \vec{dl}$ comes from the side GH. Thus,

$$\oint \vec{B} \cdot \vec{dl} = BL = \mu_0 I_{enclosed} = \mu_0 nLI$$

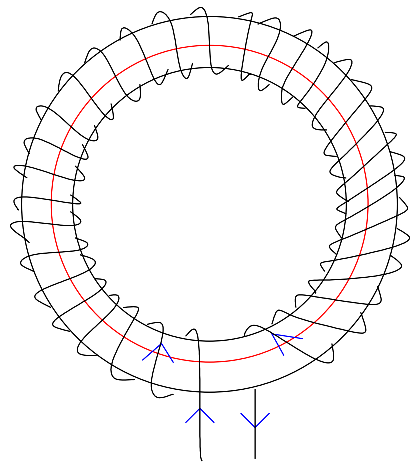
where I is the current through each turn and n is number of turns per unit length. $I_{enclosed} = nLI$ because the number of turns threading the loop is nL. Hence,

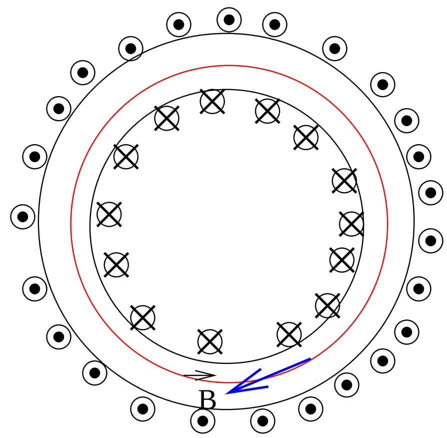
$$B = \mu_0 nI$$

is independent of the distance from the axis.

Exercise : Field of a toroid

A toroid is essentially a hollow tube bent in the form of a circle. Current carrying coils are wound over it. Use an amperian loop shown in the figure to show that the field within the toroid is $\mu_0 NI/L$, where N is the number of turns and L the circumference of the circular path.



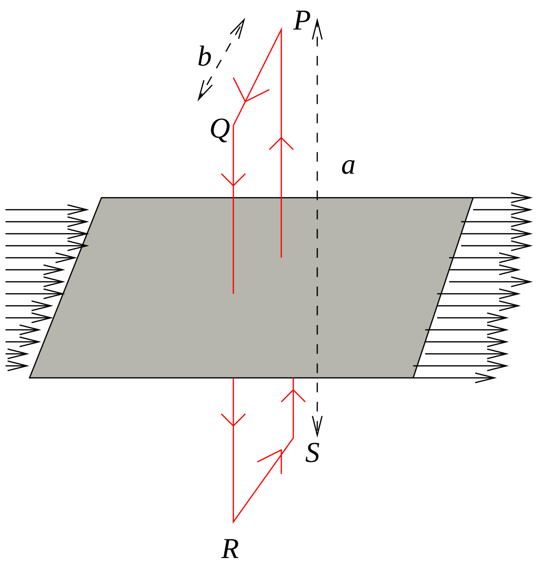


Note that as the circumference of the circular path varies with the distance of the amperian loop from the toroid axis, the magnetic field in the toroid varies over its cross section.

Take the inner radius of the toroid to be 20cm and the outer radius as 21cm. Find the percentage variation of the field over the cross section of the toroid. (Ans. 2.9%)

Example 10 : Field of an infinite current sheet

An infinite conducting sheet carries a current such that the current density is λ per unit length. Take an amperian loop as shown.



The contribution to the line integral of $\vec{B}$ from the sides QR and SP are zero as $\vec{B}$ is perpendicular to $\vec{dl}$. For PQ and RS the direction of $\vec{B}$ is parallel to the path. Hence

$$\oint \vec{B} \cdot \vec{dl} = 2bB = \mu_0 \lambda b$$

giving $B = \mu_0 \lambda / 2$.

Exercise :

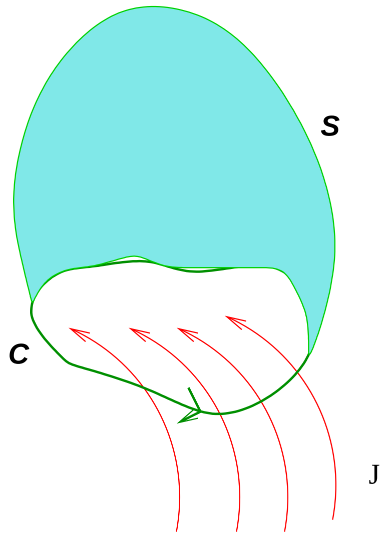
Calculate the force per unit area between two parallel infinite current sheets with current densities λ_1 and λ_2 in the same direction. (Ans.$\mu_0 \lambda_1 \lambda_2 / 2$)

Ampere's Law in Differential Form

We may express Ampere's law in a differential form by use of Stoke's theorem, according to which the line integral of a vector field is equal to the surface integral of the curl of the field,

$$\oint \vec{B} \cdot \vec{dl} = \int_S \text{curl } \vec{B} \cdot d\vec{S}$$

The surface S is *any* surface whose boundary is the closed path of integration of the line integral.



In terms of the current density $\vec{J}$, we have,

$$\int_S \vec{J} \cdot d\vec{S} = I_{encl}$$

where I_{encl} is the total current through the surface S. Thus, Ampere's law $\oint \vec{B} \cdot \vec{dl} = I_{encl}$ is equivalent to

$$\int \text{curl } \vec{B} \cdot d\vec{S} = \mu_0 \int \vec{J} \cdot d\vec{S}$$

which gives

$$\text{curl } \vec{B} = \mu_0 \vec{J}$$

You may recall that in the case of electric field, we had shown that the divergence of the field to be given by $\nabla \vec{E} = \rho/epsilon_0$. In the case of magnetic field there are no free sources (monopoles). As a result the divergence of the magnetic field is zero

$$\nabla \cdot \vec{B} = 0$$

The integral form of above is obtained by application of the divergence theorem

$$\int_S \vec{B} \cdot d\vec{S} = \int_V \nabla \cdot \vec{B} dV = 0$$

Thus the flux of the magnetic field through a closed surface is zero.

Vector Potential

For the electric field case, we had seen that it is possible to define a scalar function ϕ called the "potential" whose negative gradient is equal to the electric field $\nabla\phi = \vec{E}$. The existence of such a scalar function is a consequence of the conservative nature of the electric force. It also followed that the electric field is irrotational, i.e. $\text{curl } \vec{E} = 0$.
For the magnetic field, Ampere's law gives a non-zero curl

$$\text{curl } \vec{B} = \mu_0 \vec{J}$$

Since the curl of a gradient is always zero, we cannot express $\vec{B}$ as a gradient of a scalar function as it would then violate Ampere's law.

However, we may introduce a vector function $\vec{A}(\vec{r})$ such that

$$\vec{B} = \nabla \times \vec{A}$$

This would automatically satisfy $\nabla\cdot\vec{B} = 0$ since divergence of a curl is zero. $vecA$ is known as *vector potential*. Recall that a vector field is uniquely determined by specifying its divergence and curl. As $\vec{B}$ is a physical quantity, curl of $\vec{A}$ is also so. However, the divergence of the vector potential has no physical meaning and consequently we are at liberty to specify its divergence as per our wish. This freedom to choose a vector potential whose curl is $\vec{B}$ and whose divergence can be conveniently chosen is called by mathematicians as a choice of a *gauge*. If ψ is a scalar function any transformation of the type

$$\vec{A} \longrightarrow \vec{A} + \nabla\psi$$

gives the same magnetic field as curl of a gradient is identically zero. The transformation above is known as *gauge invariance*. (we have a similar freedom for the scalar potential ϕ of the electric field in the sense that it is determined up to an additive constant. Our most common choice of ϕ is one for which $\phi \to 0$ at infinite distances.)
A popular gauge choice for $\vec{A}$ is one in which

$$\nabla \cdot \vec{A} = 0$$

which is known as the "Coulomb gauge". It can be shown that such a choice can always be made.
Exercise :
Show that a possible choice of the vector potential for a constant magnetic field $\vec{B}$ is given by $\vec{A} = (1/2)\vec{B} \times \vec{r}$. Can you construct any other $\vec{A}$? (Hint : Take $\vec{B}$ in z-direction, express $\vec{A}$ in component form and take its curl.)

Biot-Savart's Law for Vector Potential
Biot-Savart's law for magnetic field due to a current element $\vec{dl}$

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\vec{dl}\times\hat{r}}{r^2} = -\frac{\mu_0 I}{4\pi}\vec{dl}\times\nabla(\frac{1}{r})$$

may be used to obtain an expression for the vector potential. Since the element $\vec{dl}$ does not depend on the position vector of the point at which the magnetic field is calculated, we can write

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\nabla \times (\frac{\vec{dl}}{r})$$

the change in sign is because $\nabla(\frac{\vec{dl}}{r}) = \nabla(1/r) \times \vec{dl}$.

Thus the contribution to the vector potential from the element $\vec{dl}$ is

$$d\vec{A} = \frac{\mu_0 I}{4\pi r}\vec{dl}$$

The expression is to be integrated over the path of the current to get the vector potential for the system

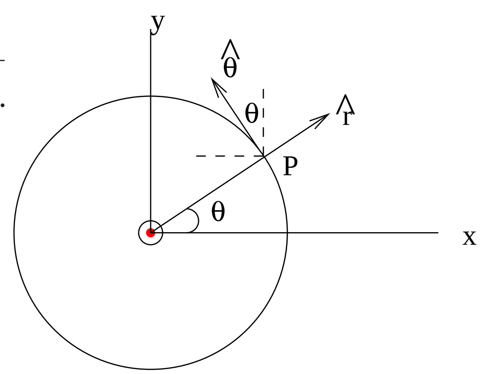
$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{\vec{dl}}{r}$$

Example 11 : Obtain an expression for the vector potential at a point due to a long current carrying wire.

Solution :

Take the wire to be along the z-direction, perpendicular to the plane of the page with current flowing in a direction out of the page. The magnitude of the field at a point P is $\mu_0 I/2\pi r$ with its direction being along the tangential unit vector $\hat{\theta}$ at P,

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$



The direction of $\vec{B}$ makes an angle $(\pi/2) + \theta$ with the x direction, where $\tan\theta = y/x$. Thus

$$\begin{aligned} \hat{\theta} &= \hat{\imath}\cos\left(\frac{\pi}{2}+\theta\right) + \hat{\jmath}\sin\left(\frac{\pi}{2}+\theta\right) \\ &= -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta \\ &= -\hat{\imath}\frac{y}{r} + \hat{\jmath}\frac{x}{r} \end{aligned}$$

Hence we have

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{(-y\hat{\imath} + x\hat{\jmath})}{x^2+y^2}$$

We wish to find a vector function $\vec{A}$ whose curl is given by the above. One can see that the following function fits the requirement

$$\vec{A} = -\frac{\mu_0 I}{4\pi}\ln(x^2+y^2)\hat{k} \tag{1}$$

In the following, we will derive this directly from the expression for Biot-Savart's law.

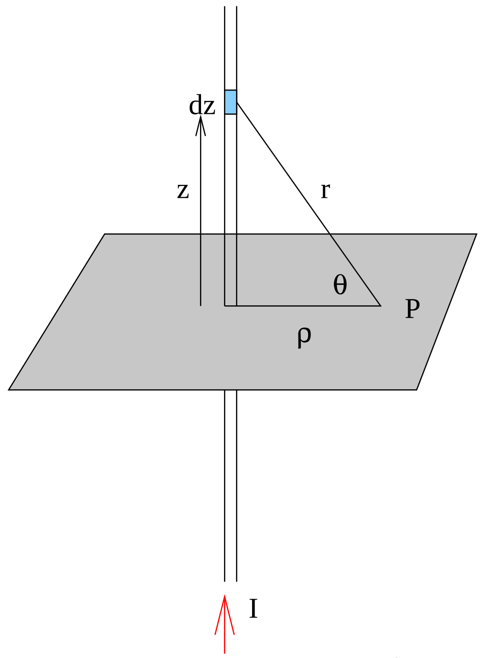
If ρ is the distance of P from an element of length dz at z of the wire, we have,

$$r^2 = x^2 + y^2 + z^2 = \rho^2 + z^2$$

Thus

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{dz}{(\rho^2 + z^2)^{1/2}}$$

If the above integral is evaluated from $z = -\infty$ to $z = +\infty$, it diverges. However, we can eliminate the infinity in the following manner. Let us take the wire to be of length $2L$ so that



$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{k}\int_{-L}^{L} \frac{dz}{(\rho^2 + z^2)^{1/2}}$$

The integral is evaluated by substituting $z = \rho\tan\theta$, so that $dz = \rho\sec^2\theta d\theta$. We get

$$\begin{aligned}\vec{A} &= \frac{\mu_0 I}{4\pi}\hat{k}\int_{-\alpha}^{\alpha}\sec\theta d\theta \\ &= \frac{\mu_0 I}{2\pi}\hat{k}\ln(\sec\alpha + \tan\alpha)\end{aligned}$$

where $\tan\alpha = L/\rho$.

In terms of L and ρ, we have

$$\sec\alpha = \frac{(L^2+\rho^2)^{1/2}}{\rho} = \frac{L}{\rho}\left(1 + \frac{\rho^2}{L^2}\right)^{1/2} \approx \frac{L}{\rho}\left(1 + \frac{\rho^2}{2L^2}\right)$$

Thus to leading order in L,

$$A = \frac{\mu_0 I}{2\pi}\hat{k}\ln(2L/\rho) = \frac{\mu_0 I}{2\pi}(\ln 2L - \ln\rho)\hat{k}$$

As expected, for $L \to \infty$, the expression diverges. However, since $\vec{A}$ itself is not physical while curl of $\vec{A}$ is, the constant term (which diverges in the limit of

$L \to \infty$) is of no consequence and $\vec{A}$ is given by

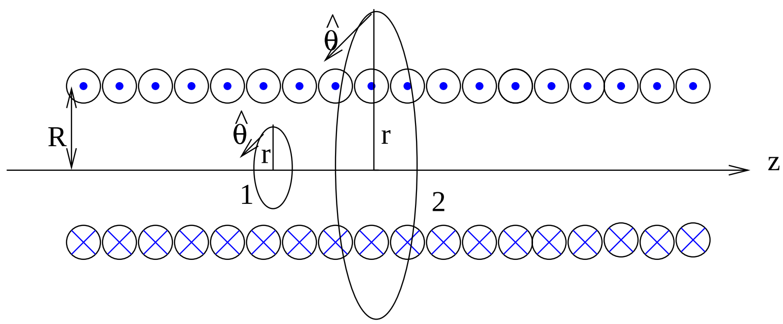
$$A = -\frac{\mu_0 I}{2\pi}\hat{k}\ln\rho = -\frac{\mu_0 I}{4\pi}\ln(x^2+y^2)$$

which is the same as Eqn. (1)

Example 12 : Obtain an expression for the vector potential of a solenoid.
Solution :
We had seen that for a solenoid, the field is parallel to the axis for points inside the solenoid and is zero outside.

$$\begin{aligned} B &= \mu_0 n I\hat{k} \text{ insidesolenoid} \\ &= 0\text{outside} \end{aligned}$$

Take a circle of radius r perpendicular to the axis of the solenoid. The flux of the magnetic field is



$$\begin{aligned} \int \vec{B}\cdot d\vec{S} &= \mu_0 n I.\pi r^2 \quad \text{for} r \le R \\ &= \mu_0 n I.\pi R^2 \quad \text{for} r \ge R \end{aligned}$$

Since $\vec{B}$ is axial, $\vec{A}$ is directed tangentially to the circle. Further, from symmetry, the magnitude of $\vec{A}$ is constant on the circumference of the circle.
Use of Stoke's theorem gives

$$\begin{aligned} \int \vec{B}\cdot d\vec{S} &= \int \text{curl } \vec{A}\cdot d\vec{S} \\ &= \oint \vec{A}\cdot d\vec{l} \\ &= \mid A \mid 2\pi r \end{aligned}$$

Thus

$$\begin{aligned}\vec{A} &= \frac{\mu_0 n I \pi r^2}{2\pi r}\hat{\theta} = \frac{\mu_0 n I r}{2}\hat{\theta} \ \text{ for} \leq R \\ &= \frac{\mu_0 n I \pi R^2}{2\pi r}\hat{\theta} = \frac{\mu_0 n I R^2}{2r}\hat{\theta} \ \text{ for} \geq R\end{aligned}$$

where $\hat{\theta}$ is the unit vector along the tangential direction.

Exercise : Obtain an expression for the vector potential inside a cylindrical wire of radius R carrying a current I. (Ans. $-\mu_0 I r^2/4\pi R^2$)

The existence of a vector potential whose curl gives the magnetic field directly gives

$$\text{div } B = 0$$

as the divergence of a curl is zero. The vector identity

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

can be used to express Ampere's law in terms of vector potential. Using a Coulomb gauge in which $\nabla \cdot \vec{A} = 0$, the Ampere's law $\nabla \times \vec{B} = -\mu_0 \vec{J}$ is equivalent to

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

which is actually a set of three equations for the components of $\vec{A}$, viz.,

$$\begin{aligned}\nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z\end{aligned}$$

which are Poisson's equations.